

**From Defects to Controllers:
Nonlinear Coarse-Graining, Phase Locking and Positional
Memory**

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Abstract

Coarse-graining a field discards its fine structure, but the product of two fine components can leave a coarse residue, the coarse-graining defect, that no coarse-grained component contains. We ask when such a defect holds usable information and when feedback can turn it into a controller of the patterns that produce it. For two fine patterns with wavenumbers n and $n + q$, the defect of their product converges, as n grows at fixed filter width, to a difference mode at wavenumber q whose phase is the relative phase of the patterns, a phase lost by every coarse-grained signal and power spectrum; in the opposite order of limits the defect vanishes, as the energy flux of a viscous Burgers shock does. When a positional template reads the difference mode and adjusts the phases of its generators, the loop is a second-order phase-locked loop: it locks whenever the detuning is at most $K_q = \kappa\chi AB/[2(\alpha + Dq^2)]$, so only long difference patterns can be held, and it is hysteretic when its signalling field is slow, below a damping of 1.193. In a translation-invariant medium the loop only pulls the frequencies together and locks at no interval of detuning. A slow bistable memory written by the difference mode can supply the missing reference. A ring of locally coupled cells reproduces the locked phase to within 0.003 rad when its integrating field spans the pattern, fails to lock when it does not, and oscillates about the set point when the loop gain exceeds the summed bandwidths of its two filters.

Introduction

Two gratings of slightly different period, laid over each other, show broad light and dark bands whose period belongs to neither grating. An observer who blurs each grating before superposing them sees two uniform grey fields and no bands; an observer who superposes first and blurs afterwards sees the bands. The difference between the two orders of operation is a coarse-graining defect: the coarse-grained product minus the product of the coarse-grained factors. The same object appears in turbulence as the subgrid stress, through which unresolved scales enter the momentum balance of the resolved ones (Leonard 1975; Germano 1992), in the reduction of many-body dynamics as the memory and noise terms of the Mori–Zwanzig formalism (Mori 1965; Zwanzig 1961; Chorin et al. 2002), and in analysis as the defect measures that record how products of weakly converging sequences fail to converge to the product of their limits (Tartar 1990; Gérard 1991; DiPerna and Majda 1987). In each of these settings the defect is treated as an error to be modelled, bounded or closed.

Living tissues suggest a different use. Several collective patterns are produced by many cells and then act on the cells that produce them. Starving *Dictyostelium* amoebae relay pulses of cyclic AMP, and the spiral waves that result organize the aggregation of the relaying cells (Sawai et al. 2005; Gregor et al. 2010). In *Bacillus subtilis* biofilms, potassium released by metabolically stressed cells depolarizes their neighbours, and the resulting waves coordinate metabolism between the interior and the periphery of the colony (Prindle et al. 2015; Liu et al. 2015). In planarians, a brief perturbation of bioelectric networks leaves animals that look normal but regenerate two heads at a constant ratio when they are cut again; the altered body plan is stored in a global pattern of resting potential, and a second perturbation resets it (Durant et al. 2017; see also Oviedo et al. 2010). Haken

described such circular causation through order parameters that emerge from the components of a system and then constrain them (Haken 1975; Haken 1983), and recent work treats multiscale regulation of this kind as collective problem-solving by cells and tissues (Levin 2022; McMillen and Levin 2024). These examples raise a question that can be posed mathematically: when does a coarse-scale signal produced by fine-scale interactions become a controller of those interactions, and what does such a controller require from the medium in which it operates?

We study it in the simplest setting in which a defect has structure: two fine periodic patterns with wavenumbers n and $n + q$, and the difference mode at wavenumber q that their product contains. The ingredients are standard, namely products of oscillations, phase locking (Adler 1946; Kuramoto 1984; Pikovsky et al. 2001), bistable memory and the energy-flux theory of turbulence; the results concern how they combine. The difference mode survives the limit $n \rightarrow \infty$ at fixed filter width and retains the relative phase of its generators, information absent from every coarse-grained individual signal and every individual power spectrum; in the opposite order of limits it vanishes, and the viscous Burgers shock shows the same non-commutation for the energy flux. The coarse-grained signals are therefore not a closed state, and the relative phase is the missing variable. When a positional template reads the difference mode and feeds it back to the phases of its generators, the loop is a second-order phase-locked loop with a closed-form locking range, a largest controllable difference wavenumber and a hysteretic regime. Without the template, in a translation-invariant medium, the loop pulls the frequencies together and locks at no interval of detuning, so a controller of this kind needs a reference that breaks the symmetry. A slow bistable memory written by the difference mode can serve as that reference. Finally, a ring of cells in which every interaction is on-site or between nearest neighbours reproduces the reduced predictions within stated limits and adds a bound on the loop gain.

Difference-frequency responses already have a biological precedent. Electric fields at frequencies too high to recruit neural firing, but differing by a frequency within the range of neural firing, evoke firing that follows their envelope at the difference frequency (Grossman et al. 2017), and a biophysical analysis argues that this requires rectification by ion channels, since passive membrane filtering cannot produce it (Mirzakhali et al. 2020). Moiré interference between much smaller grids has been proposed as the origin of the firing fields of entorhinal grid cells, which span a range of spatial scales (Blair et al. 2007). Models of bioelectric networks already show voltage memory, hysteretic switching and synchronization of bioelectric oscillations (Pietak and Levin 2017; Cervera et al. 2019). The loop studied here differs from these in the controlled variable, which is the relative phase of two fine patterns, and in the controller, which is their own difference mode. The models are minimal and no parameter is fitted to a tissue; the biological examples motivate the question and locate the prior art, and they are not evidence that the mechanism operates in any organism. A later section turns the results into experimental predictions and the controls those experiments need.

Coarse-graining defects and their limits

The defect of a product

Let P_ℓ be the Gaussian filter of width ℓ on the circle $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$, the Fourier multiplier $e^{-k^2\ell^2/2}$. For two fields u and v the defect of their product is

$$\mathcal{D}_\ell[u, v] = P_\ell(uv) - P_\ell u P_\ell v.$$

For the velocity of an incompressible fluid the same construction with $u \otimes u$ gives the subgrid stress $\tau_\ell = P_\ell(u \otimes u) - P_\ell u \otimes P_\ell u$, whose divergence appears in the filtered momentum equation (Leonard 1975; Germano 1992). Take

$$a_n(x) = A \cos(nx), \quad b_n(x) = B \cos((n+q)x + \varphi),$$

with integers $n, q \geq 1$ and a phase φ .

Proposition 1. (i) $a_n \rightharpoonup 0$ and $b_n \rightharpoonup 0$ weakly in $L^2(\mathbb{T})$, while $a_n b_n \rightharpoonup \frac{AB}{2} \cos(qx + \varphi)$. (ii) For $A = B = 1$ and every $\ell > 0$,

$$\begin{aligned} \mathcal{D}_\ell &= \frac{1}{2} e^{-q^2\ell^2/2} \cos(qx + \varphi) + \frac{1}{2} e^{-(2n+q)^2\ell^2/2} \cos((2n+q)x + \varphi) \\ &\quad - e^{-(n^2+(n+q)^2)\ell^2/2} \cos(nx) \cos((n+q)x + \varphi), \end{aligned}$$

so $\mathcal{D}_\ell \rightarrow \frac{1}{2} e^{-q^2\ell^2/2} \cos(qx + \varphi)$ uniformly as $n \rightarrow \infty$ at fixed ℓ , and $\mathcal{D}_\ell \rightarrow 0$ uniformly as $\ell \rightarrow 0$ at fixed n . (iii) The power spectra of a_n and b_n do not depend on φ , and the phase of the q -component of $\mathcal{D}_\ell$ equals φ .

Proof. The identity $2 \cos \alpha \cos \beta = \cos(\alpha - \beta) + \cos(\alpha + \beta)$ writes $a_n b_n$ as a sum of the modes q and $2n + q$; the Riemann–Lebesgue lemma gives (i), the Fourier multiplier gives (ii), and (iii) is immediate. $\square$

The relative phase φ is thus information that a coarse observer loses from each signal separately and recovers only from the defect. Its survival depends on the order of the limits and on nothing singular: every a_n is analytic, and what fails as $n \rightarrow \infty$ is the strong convergence of the sequence. If $a_n \rightarrow a$ and $b_n \rightarrow b$ strongly in L^2 , the products converge in L^1 to ab and no defect survives, so a surviving defect certifies a loss of compactness; this is the observation from which compensated compactness and H-measures start (Tartar 1990; Gérard 1991). Direct spectral filtering on 8,192 points reproduces the closed form to 9.2e-14. At $\ell = 0.1$ and $q = 1$ the defect lies within 2.6e-5 of its limit at $n = 32$ and within 5.9e-15 at $n = 64$. In the other order, at fixed $n = 64$, the largest value of the defect falls from 0.498 at $\ell = 0.1$ to 0.0041 at $\ell = 0.001$, and the small-filter expansion $\mathcal{D}_\ell \approx \ell^2 a'_n b'_n$ bounds it by $\ell^2 n(n+q) = 0.0042$ (Figure 1a–b).

The energy flux of a shock

The same non-commutation governs energy dissipation. Filtering the viscous Burgers equation $u_t + (u^2/2)_x = \nu u_{xx}$ gives $\partial_t \bar{u} + \partial_x (\bar{u}^2/2) = -\frac{1}{2} \partial_x \tau_\ell + \nu \partial_{xx} \bar{u}$ with $\bar{u} = P_\ell u$ and $\tau_\ell = P_\ell(u^2) - \bar{u}^2$, and the resolved energy passes to the unresolved scales at the local rate $\Pi_\ell = -\frac{1}{2} \tau_\ell \partial_x \bar{u}$, the one-dimensional form of the flux that Eyink (1994) and Duchon and Robert (2000) use for the Euler and Navier–Stokes equations. Consider the stationary shock $u = -\tanh(x/2\nu)$ joining $u = 1$ to $u = -1$ on the real line, an exact solution of the viscous Burgers equation (Hopf 1950; Cole 1951), filtered with the normalized Gaussian G_ℓ .

Proposition 2. (i) The viscous dissipation $\nu \int u_x^2 dx$ equals $2/3$ for every ν . (ii) For the inviscid shock $u = -\operatorname{sgn} x$, $\int \Pi_\ell dx = 2/3$ for every ℓ . (iii) For the viscous shock, $\int \Pi_\ell dx = \frac{2}{3} - \frac{2}{\sqrt{\pi}} \frac{\nu}{\ell} + O(\nu^2/\ell^2)$ as $\nu/\ell \rightarrow 0$, and $\int \Pi_\ell dx = \frac{2}{15}(\ell/\nu)^2 + O((\ell/\nu)^4)$ as $\ell/\nu \rightarrow 0$. Hence $\lim_{\ell \rightarrow 0} \lim_{\nu \rightarrow 0} \int \Pi_\ell dx = 2/3$, while $\lim_{\nu \rightarrow 0} \lim_{\ell \rightarrow 0} \int \Pi_\ell dx = 0$.

Proof. (i) follows from $\int \operatorname{sech}^4 = 4/3$. (ii) For $u = -\operatorname{sgn} x$, $\bar{u} = -\operatorname{erf}(x/\sqrt{2}\ell)$, $\partial_x \bar{u} = -2G_\ell$ and $\tau_\ell = 1 - \bar{u}^2$, so $\int \Pi_\ell dx = \int (1 - \bar{u}^2) G_\ell dx = \frac{1}{2} \int_{-1}^1 (1 - y^2) dy = 2/3$. (iii) For small ν , $P_\ell(u^2) = 1 - P_\ell \operatorname{sech}^2(x/2\nu) = 1 - 4\nu G_\ell + O(\nu^3)$ while $\bar{u}$ changes only at order ν^2 , which subtracts $4\nu \int G_\ell^2 = 2\nu/(\sqrt{\pi}\ell)$. For small ℓ , $\tau_\ell = \ell^2 u_x^2 + O(\ell^4)$, so $\int \Pi_\ell = -\frac{1}{2} \ell^2 \int u_x^3 = \frac{2}{15}(\ell/\nu)^2$, using $\int \operatorname{sech}^6 = 16/15$. $\square$

Computed by spectral filtering (Figure 1c), the flux is 0.6554 at $\nu/\ell = 0.01$ and 0.1002 at $\nu/\ell = 1$. The deficit coefficient estimated at $\nu/\ell = 0.001$ is 1.128377, against $2/\sqrt{\pi} = 1.128379$, and at $\nu/\ell = 100$ the flux is 0.99942 of its smooth asymptote.

The flux through an arbitrarily fine filter survives the inviscid limit because the limiting solution is discontinuous. This is the one-dimensional case of Onsager’s theory of ideal turbulence (Onsager 1949; Eyink and Sreenivasan 2006): solutions of the Euler equations with Hölder exponent above $1/3$ conserve energy (Constantin et al. 1994; Eyink 1994; Cheskidov et al. 2008), weak solutions that dissipate energy exist below it (Isett 2018; Buckmaster et al. 2019), and when dissipation occurs it equals the limit of the flux (Duchon and Robert 2000). The Moiré defect of Proposition 1 survives with no discontinuity in any member of its sequence. Two different things can therefore make a coarse-graining defect persist under refinement: a singular limiting solution, which carries energy across scales, and unbounded oscillation, which can carry phase. Whether the three-dimensional Euler equations develop singularities from smooth data (Beale et al. 1984; Luo and Hou 2014) bears on the first route only, and the loop studied below uses the second.

Coherence and dephasing

Consider a population of pairs with relative phases φ_j . For pairs of unit amplitude, the defect of the population average has a q -component of amplitude $\frac{1}{2} e^{-q^2 \ell^2/2} |\langle e^{i\varphi_j} \rangle|$, so the difference mode measures the coherence of the relative phase. For M pairs with independent uniform phases its mean square amplitude is $1/(4M)$, and in 2,000 simulated populations for each M from 10 to 10,000 the root-mean-square amplitude times $2\sqrt{M}$ stays within 1.0% of 1.

When the relative phases rotate at different rates the coherence decays although no pattern loses amplitude. For detunings drawn from a Cauchy distribution of width γ , $\langle e^{i\varphi_j(t)} \rangle = e^{i\varphi_0} e^{-\gamma t}$, the characteristic function of the Cauchy law; the same identity underlies the exact low-dimensional reduction of the Kuramoto model with a Lorentzian distribution of frequencies (Ott and Antonsen 2008; Strogatz 2000). The squared coherence moves into the incoherent part $1 - |\langle e^{i\varphi} \rangle|^2$, a population analogue of the split between resolved energy and defect. The loss is reversible. Conjugating every phase at a time T_e restores full coherence at $2T_e$, as in spin and plasma echoes (Hahn 1950; Gould et al. 1967), and phase mixing of the same kind is the mechanism of Landau damping (Mouhot and Villani 2011). With 10^6 pairs, $\gamma = 0.1$ and $T_e = 20$, the simulated coherence stays within $1.2e-3$ of $e^{-\gamma t}$, equals 0.1353 at T_e and returns to 1 at $2T_e$ to twelve decimals (Figure 1d). A coarse signal that depends on relative phase is therefore fragile under heterogeneity, and a system that uses it has to maintain the coherence that produces it. Phase locking does so.

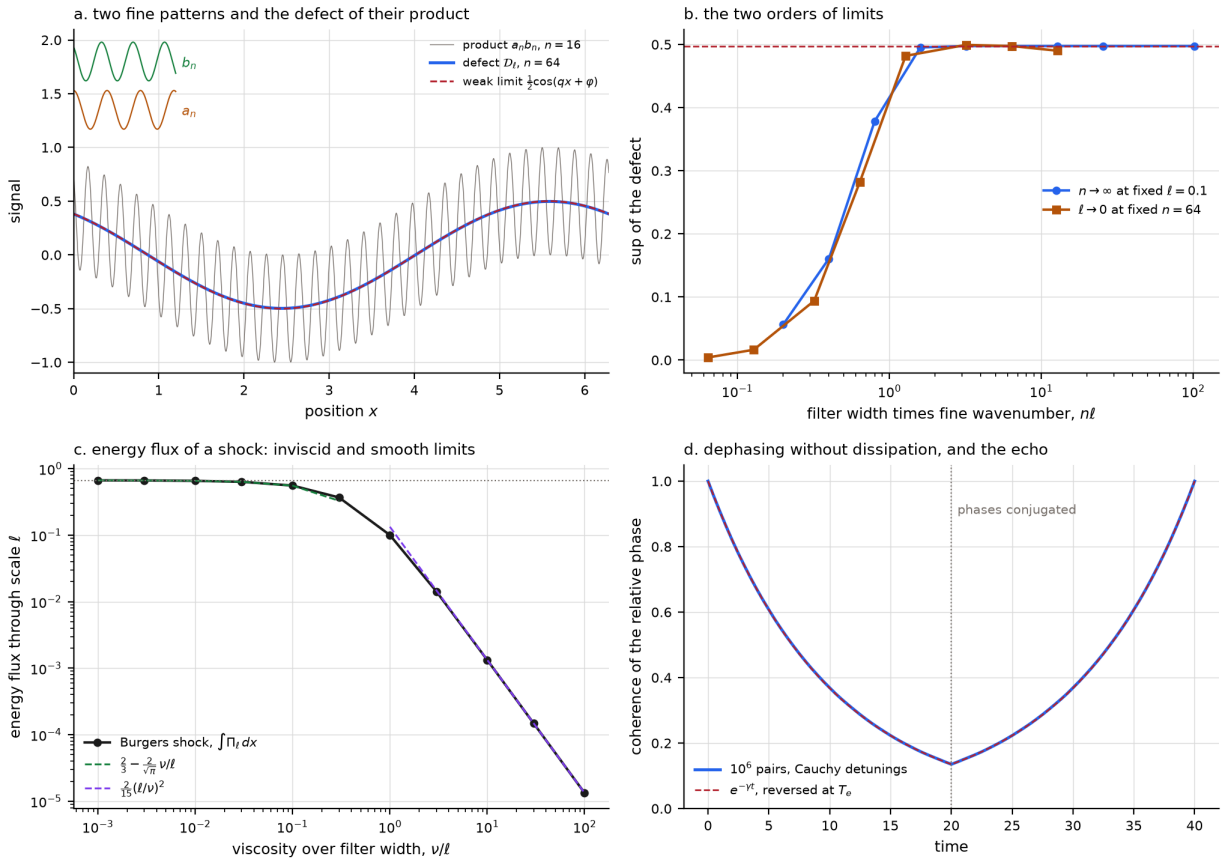


Figure 1: Coarse-graining defects under two orders of limits. (a) Two fine patterns ($n = 16$, $q = 1$, shown on a short interval), their product, and the defect at $n = 64$, $\ell = 0.1$, which coincides with the weak limit $\frac{1}{2} \cos(qx + \varphi)$ up to the filter factor. (b) The largest value of the defect against nl : it tends to its limit as n grows at fixed ℓ and to zero as ℓ shrinks at fixed n . (c) Total energy flux through the filter scale for the viscous Burgers shock, with the inviscid asymptote $\frac{2}{3} - \frac{2}{\sqrt{\pi}} \nu/\ell$ and the smooth asymptote $\frac{2}{15} (\ell/\nu)^2$. (d) Coherence of the relative phase in 10^6 pairs with Cauchy detunings ($\gamma = 0.1$), with the phases conjugated at $T_e = 20$.

Closure of the coarse description

Whether the coarse-grained signals can serve as the state of a coarse description is a question of closure: the future of the observed variables must be determined by their present values.

Proposition 3. Let $\dot{X} = F(X)$ generate a flow and let $Y = P(X)$ be an observation. If a deterministic autonomous equation $\dot{Y} = G(Y)$ holds along the observed image of every trajectory, then $DP(X)F(X)$ takes the same value at all states X with the same observation $P(X)$.

Proof. If $P(X_1) = P(X_2)$ while DPF differs at X_1 and X_2 , the observed derivatives at time zero differ, whereas $G(P(X_1)) = G(P(X_2))$. $\square$

For linear observations this is the exact-lumping condition of chemical kinetics (Li and Rabitz 1989). When it fails, the Mori–Zwanzig construction still gives an exact equation for the observed variables, at the price of a memory kernel and a noise term that represent the discarded information (Mori 1965; Zwanzig 1961; Chorin et al. 2002; Givon et al. 2004). Computational mechanics asks for the minimal state that makes the future conditionally independent of the past (Shalizi and Crutchfield 2001), and information-theoretic accounts of emergence ask when a coarse description predicts as well as the fine one (Hoel et al. 2013; Rosas et al. 2020; Barnett and Seth 2023).

For the loop defined in the next section, take as observation the coarse-grained signals $P_\ell a$ and $P_\ell b$ together with the signalling field z . Two states with $z = 0$, $\theta_1 = 0$ and relative phases 0 and $\pi/2$ have observations that differ by at most $\sqrt{2} e^{-(n+q)^2 \ell^2/2}$, which at $n = 128$ and $\ell = 0.1$ is below rounding (7.9e-15), while their values of $\partial_t z$ differ by 0.707 in the maximum norm. The coarse-grained signals and the field are therefore not a closed state. The next section shows that adding the relative phase closes the description exactly.

Phase locking by the difference mode

The loop and its exact reduction

The loop has four parts. The fine patterns are $a = A \cos(nx + \theta_1(t))$ and $b = B \cos((n+q)x + \theta_2(t))$. A signalling field is driven by their local product, $\partial_t z = D\partial_x^2 z - \alpha z + \chi ab$. A readout compares the field with a fixed template, $s(t) = \pi^{-1} \int_{\mathbb{T}} z(x, t) r(x) dx$ with $r(x) = \sin(qx)$. The readout shifts the phases of the generators in opposite directions, $\dot{\theta}_1 = \omega_1 - \frac{\kappa}{2}s$ and $\dot{\theta}_2 = \omega_2 + \frac{\kappa}{2}s$. The product stands for a nonlinear transduction step, the field for a diffusible or electrically conducted signal, and the template for any fixed spatial reference with a component at wavenumber q . Write the q -component of z as $c \cos(qx) + d \sin(qx)$, and set $\delta = \theta_2 - \theta_1$, $\Delta\omega = \omega_2 - \omega_1$, $\lambda_q = \alpha + Dq^2$ and $F = \chi AB/2$.

Proposition 4. For every $n \geq 1$,

$$\dot{\delta} = \Delta\omega + \kappa d, \quad \dot{c} = -\lambda_q c + F \cos \delta, \quad \dot{d} = -\lambda_q d - F \sin \delta,$$

and the remaining modes of z are driven only by the sum-frequency component of ab , whose amplitude in z never exceeds $F/(\alpha + D(2n + q)^2)$.

Proof. $ab = \frac{AB}{2} [\cos(qx + \delta) + \cos((2n + q)x + \theta_1 + \theta_2)]$; the Fourier modes of z evolve independently, and $s = d$ by orthogonality. $\square$

A field model that keeps all 1,024 Fourier modes of z and computes the readout by quadrature agrees with the three equations to $1.1\text{e-}14$ in δ for $n = 4, 32$ and 200 , when both are advanced by the same fourth-order exponential time-differencing scheme (Cox and Matthews 2002; Kassam and Trefethen 2005). The reduced dynamics does not depend on n : every pair of fine patterns with the same difference wavenumber produces the same coarse dynamics.

Locking range, susceptibility and scale selection

Theorem 1. Let $\kappa > 0$ and $K_q = \kappa F / \lambda_q = \kappa \chi AB / [2(\alpha + Dq^2)]$.

- (i) Equilibria exist if and only if $|\Delta\omega| \leq K_q$; they satisfy $\sin \delta^* = \Delta\omega / K_q$, $c^* = (F / \lambda_q) \cos \delta^*$ and $d^* = -(F / \lambda_q) \sin \delta^*$.
- (ii) For $|\Delta\omega| < K_q$ the equilibrium with $\cos \delta^* > 0$ is asymptotically stable and the other is a saddle.
- (iii) The locked difference pattern $z^* = (F / \lambda_q) \cos(qx + \delta^*)$ has its maximum at $x^* = -\delta^* / q$. Its phase responds to detuning with susceptibility $\partial \delta^* / \partial \Delta\omega = (K_q^2 - \Delta\omega^2)^{-1/2}$, and its slowest relaxation rate is $\sigma = \frac{1}{2}[\lambda_q - (\lambda_q^2 - 4\lambda_q K_q \cos \delta^*)^{1/2}]$ when the root is real, so that $\sigma \partial \delta^* / \partial \Delta\omega \rightarrow 1$ as $|\Delta\omega| \rightarrow K_q$.
- (iv) At a given detuning, locking is possible only for difference wavenumbers $q \leq q_c = [(\kappa F / |\Delta\omega| - \alpha) / D]^{1/2}$.

Proof. (i) Set $\dot{\delta} = 0$ and solve the linear equations for c and d . (ii) In the (δ, d) plane the Jacobian has trace $-\lambda_q$ and determinant $\lambda_q K_q \cos \delta^*$, and c decouples with eigenvalue $-\lambda_q$. (iii) Differentiate $\sin \delta^* = \Delta\omega / K_q$ and expand σ for small $\cos \delta^*$. (iv) Rearrange $K_q \geq |\Delta\omega|$. $\square$

In the overdamped parameter set ($\alpha = 1$, $D = 0.25$, $q = 2$, $\kappa = 2$, $\chi = A = B = 1$, so $\lambda_q = 2$ and $K_q = 0.5$) the relative phase stays locked at $\Delta\omega = 0.999K_q$ and drifts at 0.022 at $1.001K_q$. Relaxation rates fitted to kicked trajectories agree with σ to a relative $8.9\text{e-}5$ for $\Delta\omega / K_q$ from 0.5 to 0.999 , and the product $\sigma \partial \delta^* / \partial \Delta\omega$ is 1.038 at $0.99K_q$ and 1.011 at $0.999K_q$. Near the edge of the locking range, then, a small change of mismatch moves the coarse pattern far and the pattern takes long to settle. Beyond the edge the phase drifts at a rate within 2.9% of Adler's law $(\Delta\omega^2 - K_q^2)^{1/2}$ (Adler 1946), which is exact in the limit of a fast field. At $\Delta\omega = 0.25$ the largest lockable difference wavenumber is $q_c = 3.46$, and simulations confirm that patterns with $q \leq 3$ lock while those with $q \geq 4$ drift. Two gratings of wavenumber k_0 rotated by an angle θ have a difference wavevector of length $2k_0 \sin(\theta/2)$, so their Moiré period is the grating period divided by $2 \sin(\theta/2)$ (Amidror 2009; Bistritzer and MacDonald 2011). A two-dimensional check on a 256 by 256 grid confirms that

the steady field peaks at the difference wavevector with the predicted amplitude. With $k_0 = 20$ the loop above can lock at $\Delta\omega = 0.25$ only if the gratings are rotated by less than 9.94° .

The loop distinguishes a regulated pattern from one that is merely produced. After a kick of 0.5 rad to the relative phase, the open-loop pattern moves by $-0.5/q = -0.25$ and stays there, while with the loop closed it returns to its locked position (Figure 2d). Because the phase integrates frequency, a constant detuning is absorbed with zero error in frequency and a fixed phase offset $\arcsin(\Delta\omega/K_q)$, the integral action that the internal model principle requires for constant disturbances (Francis and Wonham 1976).

A phase-locked loop and its hysteresis

The loop is a phase-locked loop in the engineering sense: the product is the phase detector, the signalling field the loop filter, the template the reference and the generator phases the controlled oscillators (Gardner 2005). Eliminating d gives

$$\ddot{\delta} + \lambda_q \dot{\delta} + \lambda_q K_q \sin \delta = \lambda_q \Delta\omega,$$

the damped driven pendulum that also describes a Josephson junction shunted by a resistance and a capacitance (Stewart 1968; McCumber 1968). In units of time $(\lambda_q K_q)^{-1/2}$ it reads $\ddot{\theta} + \Gamma \dot{\theta} + \sin \theta = i$, with damping $\Gamma = (\lambda_q/K_q)^{1/2}$ and torque $i = \Delta\omega/K_q$. Its bifurcations are known (Levi et al. 1978; Strogatz 2018): below a critical damping Γ_c , a stable running solution coexists with the stable locked state for $i_r(\Gamma) < i < 1$. In the language of phase-locked loops the hold-in range K_q then exceeds the pull-in range $K_q i_r(\Gamma)$ (Leonov et al. 2015), and in populations of oscillators with inertia the same mechanism turns the onset of synchronization into a first-order transition with hysteresis (Tanaka et al. 1997).

We computed $i_r(\Gamma)$ by shooting along the unstable manifold of the saddle and bisecting on i until the manifold reaches the next saddle, and Γ_c by bisecting on Γ as $i \rightarrow 1$ with an extrapolation in $(1 - i)^{1/2}$. For small damping i_r agrees with the Melnikov estimate $4\Gamma/\pi$ (Guckenheimer and Holmes 1983), with ratios 0.99998 at $\Gamma = 0.01$ and 0.9975 at $\Gamma = 0.1$, and the critical damping is $\Gamma_c = 1.193$ (Figure 2c). A slow signalling field, small λ_q relative to K_q , therefore makes the loop hysteretic. In an underdamped set ($\alpha = 0.1$, $D = 0.05$, $q = 2$, $\kappa = 1.2$, giving $\lambda_q = 0.3$, $K_q = 2$ and $\Gamma = 0.387$) the pull-in boundary is $i_r = 0.475$. Adiabatic sweeps of the detuning unlock the loop at $\Delta\omega/K_q = 1.01$ on the way up and relock it only at $\Delta\omega/K_q = 0.47$ on the way down, whereas in the overdamped set the two sweeps agree to the resolution of the sweep (Figure 2b). A loop with a slow signalling field thus retains whether it was locked, a memory that needs no additional variable.

Anchoring and translation invariance

The template breaks the translation symmetry of the medium. Without it, shifting the whole configuration by s adds qs to δ and multiplies the complex amplitude $Z = c - id$ of the difference mode

by e^{iqs} , and any feedback assembled from translation-invariant local interactions has to respect this.

Theorem 2. Let $\dot{Z} = -\lambda_q Z + F e^{i\delta}$, where λ_q may be complex with positive real part, and let $\dot{\delta} = \Delta\omega + f(\delta, Z)$ with $f(\delta + qs, Z e^{iqs}) = f(\delta, Z)$ for every s . A solution with $\dot{\delta} \equiv 0$ exists only for the single detuning $\Delta\omega = -f(0, F/\lambda_q)$. In particular no interval of detuning is locked.

Proof. Invariance gives $f(\delta, Z) = g(W)$ with $W = Z e^{-i\delta}$ and $g(W) = f(0, W)$. On a solution with $\dot{\delta} \equiv 0$ the phase is constant, $W(t) \rightarrow F/\lambda_q$, and $\Delta\omega + g(W(t)) = 0$ for all t ; by continuity $\Delta\omega = -g(F/\lambda_q)$. $\square$

The theorem excludes locked states in which the relative phase is constant; it leaves open bounded oscillations of the phase with zero mean drift, which we did not observe in any of the translation-invariant simulations below. The standard translation-invariant feedback is the triad form $\dot{\delta} = \Delta\omega + \kappa \text{Im}(Z e^{-i\delta})$, which is what products of the local field with the local fine signals produce (derived in the lattice section). Its states of steady drift $\dot{\delta} = \Omega$ satisfy the pulling law

$$\Omega \left[1 + \frac{\kappa F}{\lambda_q^2 + \Omega^2} \right] = \Delta\omega,$$

so $\Omega = 0$ only at $\Delta\omega = 0$. With the parameters of the overdamped set the simulated drift follows the pulling law to 7.3e-13 for $\Delta\omega/K_q$ from 0.02 to 2 and never falls below 0.800 of the detuning, while with the template the drift is zero up to K_q (Figure 2a). For a general template whose q -component is $\text{Re}(R e^{iqx})$, the readout is $\text{Re}(Z \bar{R})$, the locking range becomes $K_q |R|$, and the locked pattern sits at a fixed offset from the template's own q -component; the rest of the template does not matter.

A difference pattern generated in a homogeneous medium therefore has a neutral position, as a Turing pattern does on a homogeneous domain (Turing 1952; Cross and Hohenberg 1993). To serve as a set point it has to be read against something that breaks the symmetry, the role that positional information plays in development (Wolpert 1969; Green and Sharpe 2015).

A stored domain as the reference

Let each position carry a slow bistable variable h , $\tau_h \dot{h} = -h + \tanh(gh + \beta z)$, with $g > 1$ and no coupling between positions. At zero drive it has stable states $\pm h^*$, and it stays bistable while the drive satisfies $|\beta z| < u_c(g) = g\sqrt{1 - 1/g} - \text{artanh} \sqrt{1 - 1/g}$, which gives $u_c = 0.5328$ and $h^* = 0.9575$ for $g = 2$.

Proposition 5. (i) A difference pattern $z = |Z| \cos(q(x - x_0))$ held long enough switches from $-h^*$ to h^* exactly the positions where $\beta |Z| \cos(q(x - x_0)) > u_c$, a domain of width $w_\infty = (2/q) \arccos(u_c/\beta |Z|)$ centred at x_0 ; a write of finite duration switches only the positions whose drive exceeds u_c by the excess at which one switch takes that long. (ii) Since $x_0 = -\delta/q$, a change $\Delta\delta$ of the relative phase alone moves the domain by $-\Delta\delta/q$; translating the pattern b alone by a

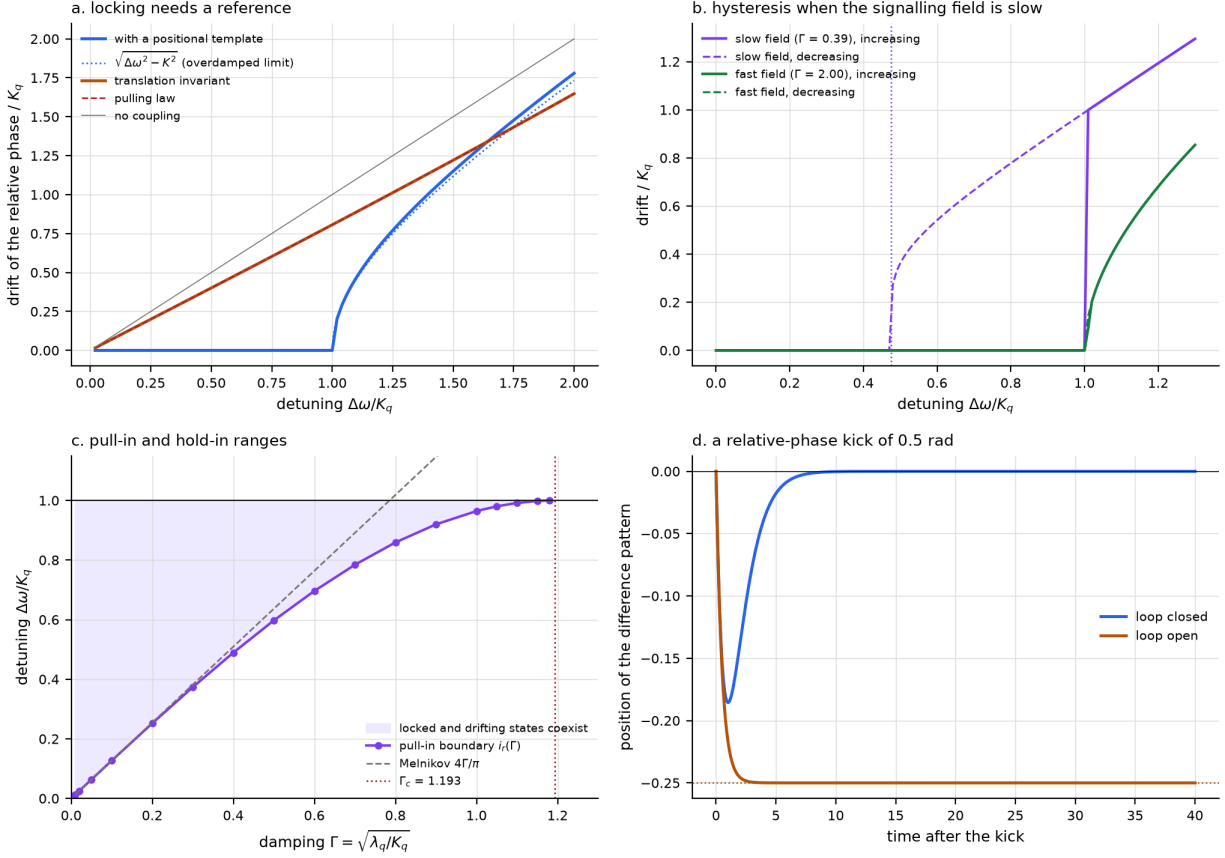


Figure 2: Locking, hysteresis and the need for a reference. (a) Drift of the relative phase against detuning, both scaled by K_q , in the overdamped set, with a positional template (locked up to K_q , then close to Adler's law) and with translation-invariant triad feedback (the pulling law, no plateau). (b) Adiabatic sweeps of the detuning upward and downward: with a slow field ($\Gamma = 0.387$) the loop unlocks at K_q and relocks near $0.47K_q$; with a fast field ($\Gamma = 2$) the sweeps coincide. (c) Pull-in boundary $i_r(\Gamma)$ of the reduced loop, the Melnikov estimate $4\Gamma/\pi$ and the critical damping Γ_c above which the hold-in and pull-in ranges coincide. (d) Position of the difference pattern after a kick of 0.5 rad to the relative phase, with the loop open and closed.

distance s changes δ by $-(n + q)s$ and moves the domain by $(n + q)s/q$, the Moiré magnification. (iii) In the reduced loop with the template $r = -\partial_x h$, a stored domain of width w centred at x_m has $|R| = (4h^*/\pi) \sin(qw/2)$ and locks the difference pattern at $x_m - \arcsin(\Delta\omega/K_q|R|)/q$ whenever $|\Delta\omega| \leq K_q|R|$; if $\beta|Z| < u_c$, no stored position changes state.

Proof. (i) Each position is an independent bistable switch whose lower state disappears when its drive exceeds u_c , the maximum of $\text{artanh } h - gh$ on the lower branch; below u_c both states persist. (ii) follows from $x_0 = -\delta/q$. (iii) The q -component of a domain of height $2h^*$ and width w has amplitude $4h^* \sin(qw/2)/(\pi q)$; differentiation multiplies it by q and shifts its phase so that, by the general-template form of Theorem 1, the stable locked pattern is centred on the domain at zero detuning. $\square$

In the simulations the ring has 256 positions, the fine patterns are rigid, the readout is the quadrature of the reduced loop, and $n = 20$, $q = 1$, $\alpha = 0.5$, $D = 0.1$, $g = 2$, $\beta = 1$ and $\tau_h = 10$. An input of amplitude $A = B = 1$ gives a peak drive $\beta|Z| = 0.833$ and, after a write of duration 300, a domain of width 1.694, against 1.715 predicted for this duration and 1.754 for an unlimited write. Over amplitudes from 0.85 to 1.5 the prediction for the actual duration stays within 0.032 of the simulated width, which the grid resolves only in steps of 0.025. Advancing the relative phase by 1 rad moves the domain's centre to -0.994 , against -1 predicted. With $n = 20$ the magnification is 21: translating b by 0.048, about a sixth of its own wavelength, moves the stored domain by 1, a sixth of the ring (Figure 3a–b).

To test reading, two rings store domains centred at 0 and at π and then receive identical inputs of amplitude 0.6, below the write threshold ($\beta|Z| = 0.300$), starting from a difference pattern at $-\pi/2$ with the loop closed ($\kappa = 2$). Each pattern ends within $1e-6$ rad of its ring's stored domain, and no stored position changes state (Figure 3c). An input of amplitude 1 at the antipode with the loop open erases the stored domain and writes a new one centred at π .

The two rings realize the closure failure of Proposition 3 in operational form. They have the same fine inputs and the same initial field, their futures differ, and the difference is held in h . The resemblance to the cryptic planarian phenotype, in which a normal appearance hides a stored bioelectric state that decides the outcome of the next cut (Durant et al. 2017), is structural only, and nothing here models regeneration. Models of bioelectric networks already produce voltage memory and hysteretic switching (Pietak and Levin 2017); the loop adds a way for such a memory to act as the positional reference that Theorem 2 requires, so that a pattern generated by the tissue returns to where an earlier pattern was.

A ring of locally coupled cells

The reduced loop assumes rigid fine patterns and a readout that integrates over the whole domain. A lattice relaxes both assumptions. Each of the 256 cells of a ring holds two phase oscillators, $\psi_{1,j}$ and $\psi_{2,j}$, each coupled to its counterpart in the two neighbouring cells by $K_c[\sin(\psi_{j+1} - \psi_j) + \sin(\psi_{j-1} - \psi_j)]$ with $K_c = 2$; the fine patterns are twisted states of winding numbers n and $n + q$ (Wiley et al. 2006), and cell j emits $a_j = A \cos \psi_{1,j}$ and $b_j = B \cos \psi_{2,j}$. The signalling field obeys

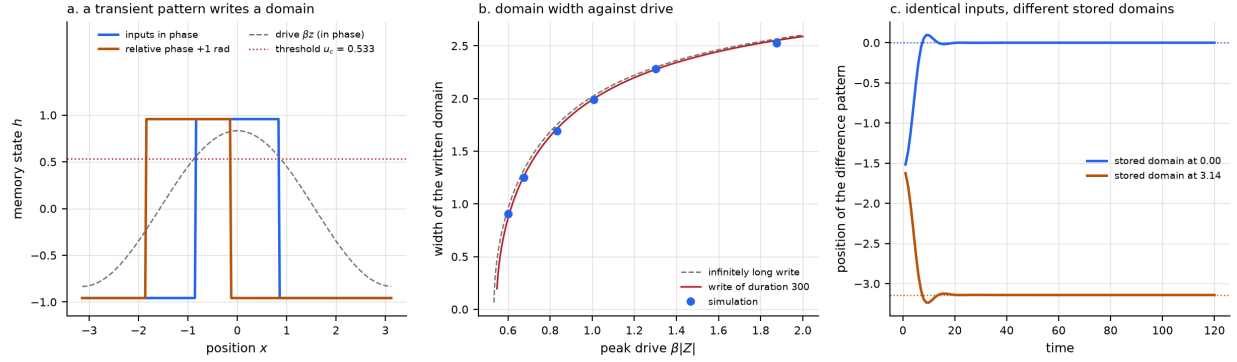


Figure 3: A difference pattern writes a positional memory that anchors later patterns. (a) Memory state after a write of duration 300 at peak drive 0.833, for two inputs that differ only in relative phase (by 1 rad), with the drive during writing and the switching threshold u_c . (b) Width of the written domain against peak drive: simulation, the prediction for the write duration, and the limit of an unlimited write. (c) Position of the difference pattern in two rings that store domains at 0 and at π and receive identical inputs below the write threshold, with the loop closed.

the discrete diffusion equation driven by the local product $a_j b_j$. The integration of the readout is done by a second field y with decay μ and fast diffusion D_y , driven by the local product $\rho z_j r_j$ with a template value $r_j = \sin(qx_j)$ stored in each cell, and each cell shifts its two frequencies by $-\kappa_1 y_j$ and $+\kappa_2 y_j$. Every interaction is on-site or between nearest neighbours. The mean of y obeys $\dot{\bar{y}} = -\mu \bar{y} + \rho d/2$ exactly, so the reduced loop has $\kappa = (\kappa_1 + \kappa_2)\rho/(2\mu)$ and gains a third variable. Its linearization has characteristic polynomial

$$r^3 + (\lambda_q + \mu)r^2 + \lambda_q \mu r + \lambda_q \mu K_q \cos \delta^* = 0,$$

and the Routh–Hurwitz conditions make the locked state stable if and only if $K_q \cos \delta^* < \lambda_q + \mu$: the loop gain is bounded by the sum of the bandwidths of its two filters, and above the bound the locked state loses stability through a Hopf bifurcation at frequency $(\lambda_q \mu)^{1/2}$.

With $\alpha = 1$, $D_z = 0.25$, $q = 2$, $\mu = 5$, $\rho = 1$ and $\kappa_1 = \kappa_2 = 10$, the reduced constants are $\lambda_q = 2.000$ and $K_q = 0.500$. With $D_y = 200$ the locked relative phase on the lattice differs from $\arcsin(\Delta\omega/K_q)$ by at most 0.003 rad over $n = 8, 16$ and 32 and $\Delta\omega/K_q = 0, 0.5$ and 0.9 , and the three fine wavenumbers give locked phases within 0.0013 rad of one another. The lattice is locked at $0.97K_q$ and drifts at $1.03K_q$, and after a kick of 0.2 rad at $\Delta\omega = 0.9K_q$ it relaxes at 0.256, within 3.4% of the slowest root of the cubic.

The integrating field has to span the pattern (Figure 4b). At $\Delta\omega = 0.5K_q$ the lattice fails to lock when the space constant $(D_y/\mu)^{1/2}$ of y is 0.32 or 0.63; it locks with a phase error of 0.25 rad at 1.0, 0.033 at 2.0, $4.9\text{e-}4$ at 6.3 and $2.0\text{e-}5$ at 14.1, for a difference pattern of wavelength π . A frequency shift that varies across the difference wavelength bends the twisted states, and nearest-neighbour coupling resists bending at that wavelength only weakly: the relaxation rate of a bend at wavenumber $2q$ is 0.0174.

The gain bound appears as predicted (Figure 4c). With a slow integrator ($\mu = 0.5$) and $\kappa_1 = \kappa_2 = 6$, so $K_q = 3.000$ against $\lambda_q + \mu = 2.500$, the lattice leaves the locked state at $\Delta\omega = 0$ and settles into an oscillation of the relative phase with peak-to-peak amplitude 2.44 rad about the set point, against 2.39 rad in the reduced four-variable loop, while at $\Delta\omega = 0.98K_q$, where $K_q \cos \delta^* = 0.60$, it stays locked.

Removing the template gives a translation-invariant lattice. The integrators then read products of local quantities only, $z_j b_j \sin \psi_{1,j}$ and $z_j a_j \sin \psi_{2,j}$, whose averages reduce to the triad form of the previous section with $\kappa = (\kappa_1 B + \kappa_2 A)\rho/(4\mu)$. At moderate gain ($\kappa_1 = \kappa_2 = 10$, $D_y = 1000$) the drift is between 0.81 and 0.89 of the detuning for detunings from 0.005 to 0.1, close to the pulling law's 0.889. At high gain ($\kappa_1 = \kappa_2 = 40$, $D_y = 200$) the drift falls to 0.093 of the detuning at the smallest detunings and rises to 0.58 at a detuning of 0.1, below the pulling law's 0.667 throughout. In every run the drift stays positive and, at small detuning, proportional to the detuning (Figure 4d). At high gain the loop bends the soft twisted states, the bend moves with the difference pattern and relaxes slowly, and the pattern has to drag it. Theorem 2 does not cover this regime, since the bend is an additional variable, but the whole lattice is still translation-invariant, and the bend moves with the pattern it would have to anchor.

Relation to scale interactions in turbulence

The difference-mode loop has the geometry of a nonlocal triad: two fine modes at wavenumbers n and $n + q$ and a coarse mode at q , which together form a resonant triad. In three-dimensional turbulence such triads are responsible for much of the interaction between scales while the energy they move stays local, passing between the two fine modes (Domaradzki and Rogallo 1990), with comparatively little feedback on the large scale (Waleffe 1992). The loop uses the same triad with the roles reversed: the energy of the coarse mode is irrelevant, and its phase is read and fed back. A turbulent flow has no template and nothing that reads phases, so the parallel shows only that the algebra of this controller is present wherever quadratic nonlinearities couple scales. It gives no reason to expect turbulent flows to regulate themselves in this way.

Experimental predictions

The results translate into predictions for any preparation in which two fine spatial patterns can be imposed and a coarse response measured, for example membrane voltage set with optogenetic reagents (Adams et al. 2014) under patterned illumination and read by a reporter downstream of a rectifying step.

- **Difference wavevector.** Two imposed patterns with wavevectors $\mathbf{k}_1$ and $\mathbf{k}_2$ produce a coarse response at $\mathbf{k}_1 - \mathbf{k}_2$; for equal gratings rotated by θ its period is the grating period divided by $2 \sin(\theta/2)$.
- **Magnification.** Translating one input by s moves the coarse response by $(n + q)s/q$, and translating both moves it by s . Direct sensing of either input predicts shifts of order s .

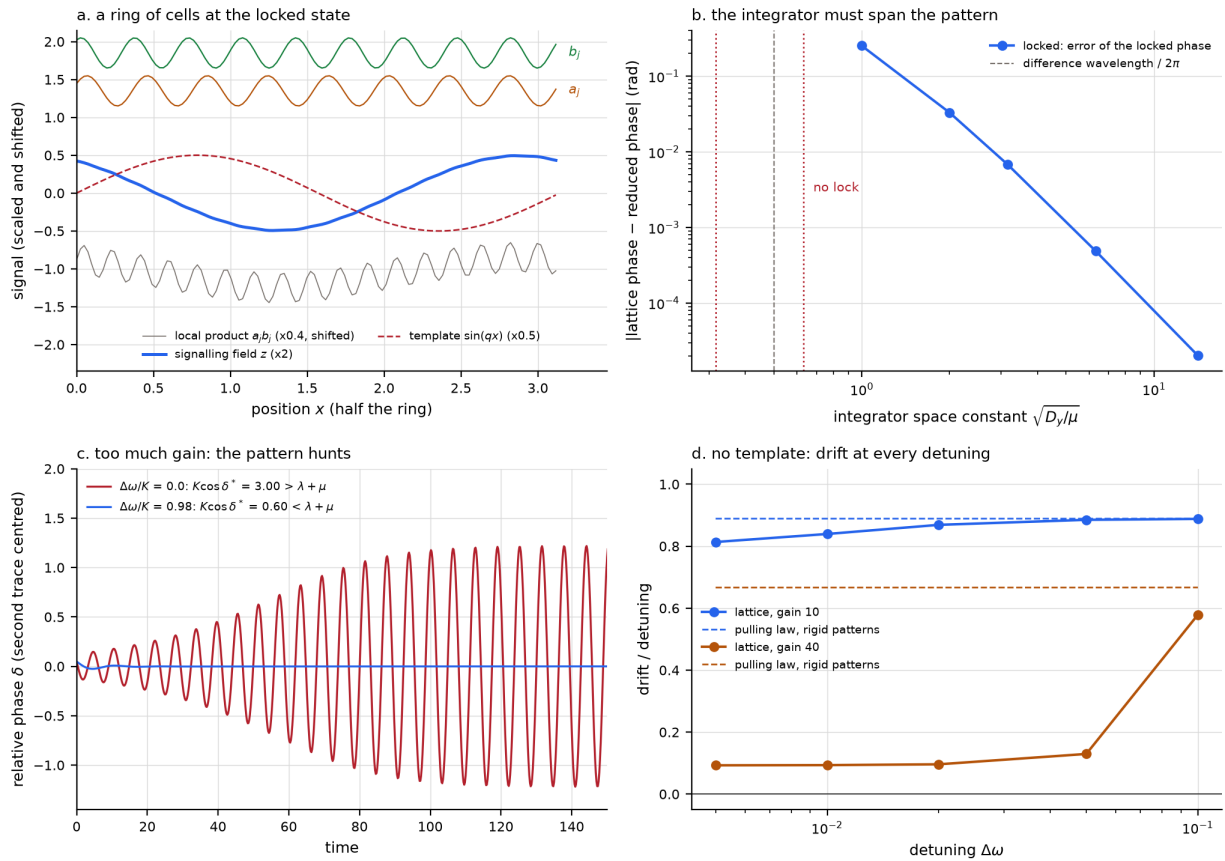


Figure 4: A ring of 256 locally coupled cells. (a) Half of the ring at the locked state ($n = 16$, $q = 2$, $\Delta\omega = 0.5K_q$): the two cell signals, their local product, the signalling field and the template. (b) Difference between the locked phase on the lattice and the reduced prediction against the space constant of the integrating field; dotted lines mark space constants at which the lattice does not lock, and the dashed line marks the difference wavelength divided by 2π . (c) Relative phase with a slow integrator and high gain: beyond the gain bound at zero detuning the pattern oscillates about its set point, and at $\Delta\omega = 0.98K_q$ it stays locked. (d) Drift divided by detuning without a template, for two gains, with the pulling law for rigid patterns.

- **Phase at fixed power.** Changing only the relative phase of the inputs moves the coarse response. Any account that uses the power spectra of the inputs predicts no change.
- **Locking range.** With feedback, the largest tolerated frequency mismatch falls as $1/(\alpha + Dq^2)$, and for rotated gratings locking fails beyond a critical angle.
- **Critical slowing.** Near the edge of the locking range, the recovery rate after a phase perturbation and the inverse of the positional susceptibility both vanish as $(K_q^2 - \Delta\omega^2)^{1/2}$.
- **Hysteresis.** If the signalling field relaxes slowly ($\Gamma < 1.193$), the mismatch at which the response unlocks exceeds the mismatch at which it relocks.
- **Reference.** In a preparation with no spatial reference at wavenumber q , the coarse response drifts at a rate proportional to the mismatch; adding a landmark, prepatter or boundary with a component at q produces a plateau of zero drift.
- **Memory.** A transient phase-only change of the inputs writes a persistent state displaced by the Moiré magnification; preparations with different stored states send identical later inputs to different positions; below a drive threshold the stored state captures the pattern and above it the pattern overwrites the state.

Each prediction needs controls. A fluorescent reporter or imaging system can generate difference frequencies of its own, so a coarse response has to be confirmed with an independent linear readout and by perturbing the proposed transduction step; in temporal interference stimulation the analogous confound, rectification outside the target and in the measurement, led to a comparable list of controls (Mirzakhali et al. 2020). Randomizing the relative phase while preserving the mean stimulation removes the coherence on which the difference mode depends, and imposing the coarse pattern directly at wavenumber q bypasses the transduction. Two outcomes would count against the mechanism: a coarse response that follows $\mathbf{k}_1 - \mathbf{k}_2$ but whose blockade leaves the collective outcome unchanged would show a pattern without a controller, and a reaction–diffusion model without phase dependence that reproduces the position shifts would make the difference mode redundant.

What the loop establishes about regulation

The loop meets the operational criteria of a controller: it has a set point, the position fixed by the template or by a stored domain; an error signal, the readout; an actuator, the phase shift of the generators; and it rejects disturbances at a computable rate. Its coarse variable is multiply realizable in the sense of Proposition 4, since the reduced dynamics does not depend on n and the lattice gives the same locked phase for three fine wavenumbers. It is causally effective under intervention, since a phase-only change moves the coarse pattern and, with memory, the stored state. Its feedback restores the collective outcome after a perturbation. These are the properties through which multiscale competency and basal agency are defined operationally (Pezzulo and Levin 2016; Levin 2022; McMillen and Levin 2024), and the model shows how they can arise from a coarse-graining defect. They concern regulation. Whether regulation of any kind is accompanied by experience is a separate question, on which the leading neuroscientific theories of consciousness

are still under direct adversarial test (Cogitate Consortium et al. 2025), and the loop bears on it in no way.

Limitations

The theorems assume rigid fine patterns. On the lattice soft patterns bend, and the bend adds co-moving variables that Theorem 2 does not cover; the simulations show no locking in that regime, which supports the conclusion of the theorem without proving it. The lattice is one-dimensional, and only the difference-wavevector law was checked in two dimensions. Phase oscillators stand in for oscillating membrane potentials and a product for a rectifying transduction, and no parameter comes from a tissue. The loop contains no noise; noise would turn the edge of the locking range into a rate of phase slips, as for a particle escaping over the barriers of a tilted periodic potential. The template is given, and only the memory section shows one way it could arise. The experiments are proposals, and the bioelectric literature cited here documents collective regulation without documenting difference modes (Levin 2014, 2021).

Conclusion

A coarse-graining defect becomes a controller when three conditions hold together. The defect contains a variable that the coarse description otherwise lacks, here the relative phase of two fine patterns. A loop reads that variable against a reference and feeds it back to the generators, which makes it a phase-locked loop with a locking range set by the wavenumber of the difference mode. And the reference breaks the translation symmetry that otherwise leaves the variable neutral, a role that a memory written by the loop's own earlier activity can fill. Each condition can fail separately in an experiment, and the predictions above test them in that order: first whether the difference mode exists and follows the relative phase, then whether feedback locks it, then whether locking depends on a reference.

Reproducibility

The simulation in `simulation/` regenerates every number and figure: `uv run run_all.py` writes `output/results.json` and the four figures and evaluates 37 internal checks (closed forms against computation, reductions against field models, predicted boundaries against simulation). The claims ledger `claims.yaml` binds each reported number to its location in `results.json` through a recorded execution.

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